

Auto-Bäcklund Transformation and Solitary-wave Solutions to Non-integrable Generalized Fifth-order Nonlinear Evolution Equations

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We show that the application of the truncated Painlevé expansion and symbolic computation leads to a new class of analytical solitary-wave solutions to the general fifth-order nonlinear evolution equations which include Lax, Sawada-Kotera (SK), Kaup-Kupershmidt (KK), and Ito equations. Some explicit solitary-wave solutions are presented.

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While it is easy to write down in closed form a solitary wave solution for the simplest standard model, namely the Korteweg-de Vries (KdV) equation, it has proved quite difficult to obtain such solutions for the problems from which the KdV equation was derived as a First approximation [1]. As such KdV hierarchy models, we investigate the generalized non-integrable fifth-order nonlinear evolution equations of the form [2]

$$u_t + \alpha uu_{xxx} + \beta u_x u_{xx} + \gamma u^2 u_x + u_{xxxxx} = 0, \quad (1)$$

where α, β , and γ are constant model parameters. This model includes the Lax [3], Kaup-Kupershmidt (KK) [4 - 7], Sawada-Kotera (SK) [9], and Ito equations [8]. As the constants α, β , and γ take different values, the properties of (1) drastically change. For instance, the Lax equation with $\alpha = 10, \beta = 20$, and $\gamma = 30$, and the SK equation where $\alpha = \beta = \gamma = 5$, are completely integrable. These two equations have N -soliton solutions and an infinite set of conservation laws. The KK equation, with $\alpha = 10, \beta = 25$, and $\gamma = 20$, is also known to be integrable [6] and to have bilinear representations [10, 11], but the explicit form of its N -soliton solution is apparently not known. A

fourth equation in the model is the Ito equation, with $\alpha = 3, \beta = 6$, and $\gamma = 2$, which is not completely integrable but has a limited number of conservation laws [8]. More recently, using a simplified version of Hirota's method, Hereman and Nuseir [2] explicitly constructed multi-soliton solutions of the KK equation for which soliton solutions were not previously known. However, to our knowledge no attempt has been made for finding more general solitary-wave solutions other than the above mentioned models because of lengthy and nearly impossible calculations without proper symbolic computation packages. By utilizing the symbolic package *Maple*, Hong [12] recently found some analytical solitary-wave solutions to the general fifth-order water models in [13], with some constraints on the model parameters.

In this work, we make use of both the truncated Painlevé expansion and the symbolic computation method [14 - 17] to obtain an *auto-Bäcklund transformation* and certain explicit solitary-wave solutions for the generalized non-integrable fifth-order nonlinear evolution equation, which are different from the solutions of the models mentioned above [3 - 8].

A non-linear partial differential equation (NPDE) is said to possess the Painlevé property when the

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solutions of the NPDE are “single valued” about the movable, singularity manifold which is “non-characteristic”. To be more precise, if the singularity manifold is determined by

$$\phi(z_1, z_2, \dots, z_n) = 0, \quad (2)$$

and $u = u(z_1, z_2, \dots, z_n)$ is a solution of the NPD, then it is required that

$$u = \phi^\alpha \sum_{j=0}^{\infty} u_j \phi^j, \quad (3)$$

where $u_0 \neq 0$, $\phi = \phi(z_1, z_2, \dots, z_n)$, $u_j = u_j(z_1, z_2, \dots, z_n)$ are analytical function of (z_j) in the neighborhood of the manifold (3), and α is a negative, rational number. Substitution of (3) into the NPDE determines the allowed values of α and defines the recursion relation for u_j , $j = 0, 1, 2, \dots$. When the ansatz (3) is correct, the NPDE is said to possess the Painlevé property and is conjectured to be integrable [14, 17].

In order to find solitonic solutions for (1), we truncate the Painlevé expansion, (3), at the constant-level term in the senses of Tian and Gao [15] and Khater et al. [17],

$$u(x, t) = \phi^{-J}(x, t) \sum_{l=0}^J u_l(x, t) \phi^l(x, t). \quad (4)$$

On balancing the highest-order contributions from the linear term (i.e. u_{xxxxx}) with the highest order contributions from the nonlinear term (i.e. $u^2 u_x$), we get $J = 2$, so that

$$u(x, t) = \frac{u_0(x, t)}{\phi^2(x, t)} + \frac{u_1(x, t)}{\phi(x, t)} + u_2(x, t). \quad (5)$$

We will stay with the general assumption that $\phi_x \neq 0$ but will not initially impose any constraints on the model parameters α, β, γ . When substituting the above expressions into (1) with the symbolic program package *Maple*, we let the coefficients of like powers of ϕ vanish, so as to get the set of Painlevé-Bäcklund (PB) equations,

$$\phi^{-7} : -720 u_0 \phi_x^5 - 12 \beta u_0^2 \phi_x^3 - 24 \alpha u_0^2 \phi_x^3 - 2 \gamma u_0^3 \phi_x = 0, \quad (6)$$

$$\begin{aligned} \phi^{-6} : & -30 \alpha u_1 u_0 \phi_x^3 + 18 \alpha u_0^2 \phi_x \phi_{xx} + 18 \alpha u_0 \phi_x^2 u_{0,x} + 1200 u_0 \phi_x^3 \phi_{xx} - 10 \beta u_0 \phi_x^3 u_1 - 4 \beta u_0^2 \phi_x \phi_{xx} \\ & + 14 \beta u_0 \phi_x^2 u_{0,x} - 120 u_1 \phi_x^5 - 5 \gamma u_0^2 u_1 \phi_x + \gamma u_0^2 u_{0,x} + 600 \phi_x^4 u_{0,x} = 0, \end{aligned} \quad (7)$$

$$\begin{aligned} \phi^{-5} : & -6 \alpha u_0 \phi_x u_{0,xx} - 4 \gamma u_0^2 \phi_x u_2 - 24 \alpha u_2 u_0 \phi_x^3 + 6 \beta u_1 \phi_x^2 u_{0,x} - 2 \beta u_1^2 \phi_x^3 + 4 \beta u_0 \phi_x u_1 \phi_{xx} \\ & + 18 \alpha u_1 \phi_x^2 u_{0,x} + 24 \alpha u_1 u_0 \phi_x \phi_{xx} + 2 \gamma u_0 u_1 u_{0,x} - 4 \gamma u_0 u_1^2 \phi_x + 6 \alpha u_0 \phi_x^2 u_{1,x} + 10 \beta u_{1,x} u_0 \phi_x^2 \\ & - 2 \beta u_0 \phi_x u_{0,xx} - 6 \alpha u_0 \phi_{xx} u_{0,x} - 2 \beta u_{0,x} u_0 \phi_{xx} - 4 \beta \phi_x u_{0,x}^2 - 720 \phi_x^2 \phi_{xx} u_{0,x} + 120 \phi_x^4 u_{1,x} \\ & + 240 u_1 \phi_x^3 \phi_{xx} + \gamma u_0^2 u_{1,x} - 240 u_0 \phi_x^2 \phi_{xxx} - 360 u_0 \phi_x \phi_{xx}^2 - 2 \alpha u_0^2 \phi_{xxx} - 6 \alpha u_1^2 \phi_x^3 \\ & - 240 \phi_x^3 u_{0,xx} = 0, \end{aligned} \quad (8)$$

$$\begin{aligned} \phi^{-4} : & \gamma u_1^2 u_{0,x} - \gamma u_1^3 \phi_x + 90 \phi_{xx}^2 u_{0,x} + 60 \phi_x^2 u_{0,xxx} + \beta u_{0,x} u_{0,xx} - 6 \alpha u_2 u_1 \phi_x^3 + 18 \alpha u_2 u_0 \phi_x \phi_{xx} \\ & + 2 \gamma u_0 u_2 u_{0,x} - 3 \alpha u_0 u_1 \phi_{xxx} + 2 \gamma u_0 u_1 u_{1,x} + 6 \alpha u_1 \phi_x^2 u_{1,x} + 4 \beta u_1 \phi_x^2 u_{1,x} + 18 \alpha u_2 \phi_x^2 u_{0,x} \\ & + 6 \alpha u_1 \phi_x u_{0,xx} - \beta u_1 \phi_x u_{0,xx} - \beta u_{0,x} u_1 \phi_{xx} - 6 \alpha u_1 \phi_{xx} u_{0,x} + 6 \alpha u_1^2 \phi_x \phi_{xx} + \beta u_1^2 \phi_x \phi_{xx} \\ & - 6 \gamma u_1 u_2 u_0 \phi_x - 2 \beta u_0 \phi_x u_{1,xx} + 6 \beta u_{2,x} u_0 \phi_x^2 - 2 \beta u_{1,x} u_0 \phi_{xx} - 3 \alpha u_0 u_{1,x} \phi_{xx} - 3 \alpha u_0 u_{1,xx} \phi_x \\ & + 180 \phi_x \phi_{xx} u_{0,xx} - 6 \beta u_{0,x} u_{1,x} \phi_x - 60 \phi_x^3 u_{1,xx} + 120 \phi_x \phi_{xxx} u_{0,x} + \gamma u_0^2 u_{2,x} - 60 u_0 \phi_{xx} \phi_{xxx} \\ & + \alpha u_0 u_{0,xxx} - 180 \phi_x^2 u_{1,x} \phi_{xx} - 60 u_1 \phi_x^2 \phi_{xxx} - 90 u_1 \phi_x \phi_{xx}^2 + 30 u_0 \phi_x \phi_{xxx} = 0, \end{aligned} \quad (9)$$

$$\begin{aligned}
\phi^{-3} : & \alpha u_0 u_{1,xxx} + \gamma u_{1,x} u_1^2 - 2\beta u_{2,x} u_0 \phi_{xx} - 2\beta u_0 \phi_x u_{2,xx} + 2\gamma u_0 u_1 u_{2,x} - 2u_0 \phi_t - 2u_0 \phi_{xxxxx} \\
& - 4\beta u_{2,x} \phi_x u_{0,x} + 20u_{1,xxx} \phi_x^2 + 30\phi_{xx}^2 u_{1,x} - 20\phi_{xx} u_{0,xxx} - 10\phi_x u_{0,xxx} - 10\phi_{xxx} u_{0,x} \\
& - 20\phi_{xxx} u_{0,xx} + 6\alpha u_2 u_1 \phi_x \phi_{xx} + 2\gamma u_1 u_2 u_{0,x} + 2\beta u_{2,x} u_1 \phi_x^2 - 3\alpha u_1 u_{1,xx} \phi_x - 3\alpha u_1 u_{1,x} \phi_{xx} \\
& - \beta u_{1,x} u_1 \phi_{xx} - \beta u_{1,xx} u_1 \phi_x + 2\gamma u_{1,x} u_0 u_2 - 2\alpha u_2 u_0 \phi_{xxx} - 2\gamma u_2^2 u_0 \phi_x + 6\alpha u_2 \phi_x^2 u_{1,x} \\
& - 6\alpha u_2 \phi_x u_{0,xx} - 6\alpha u_2 \phi_{xx} u_{0,x} - 2\gamma u_1^2 u_2 \phi_x + 40\phi_x u_{1,x} \phi_{xxx} + \alpha u_1 u_{0,xxx} - 2\beta u_{1,x}^2 \phi_x \\
& + \beta u_{1,xx} u_{0,x} + \beta u_{1,x} u_{0,xx} + 60\phi_x u_{1,xx} \phi_{xx} + 10u_1 \phi_x \phi_{xxx} + 20u_1 \phi_{xx} \phi_{xxx} - \alpha u_1^2 \phi_{xxx} \\
& + 10\alpha \phi_x^2 u_{1,xxx} - 3\mu \phi_x u_{0,xx} - 5\alpha \phi_x u_{0,xxx} - 4\gamma u_{1,x}^2 \phi_x + 3\mu \phi_x^2 u_{1,x} = 0, \quad (10)
\end{aligned}$$

$$\begin{aligned}
\phi^{-2} : & -u_1 \phi_t + \alpha u_2 u_{0,xxx} - u_1 \phi_{xxxxx} - 2\beta u_{2,x} u_{1,x} \phi_x + 2\gamma u_1 u_2 u_{1,x} - \gamma u_2^2 u_1 \phi_x - \alpha u_2 u_1 \phi_{xxx} \\
& + 2\gamma u_{2,x} u_0 u_2 - 3\alpha u_2 u_{1,x} \phi_{xx} - 3\alpha u_2 u_{1,xx} \phi_x - \beta u_{2,xx} u_1 \phi_x - \beta u_{2,x} u_1 \phi_{xx} + u_{0,xxxxx} + u_{0,t} \\
& + \beta u_{1,x} u_{1,xx} - 10u_{1,xx} \phi_{xxx} - 5u_{1,x} \phi_{xxxx} + \gamma u_2^2 u_{0,x} + \gamma u_{2,x} u_1^2 + \alpha u_1 u_{1,xxx} - 5u_{1,xxx} \phi_x \\
& + \alpha u_0 u_{2,xxx} - 10u_{1,xxx} \phi_{xx} + \beta u_{2,x} u_{0,xx} + \beta u_{2,xx} u_{0,x} = 0, \quad (11)
\end{aligned}$$

$$\phi^{-1} : \alpha u_1 u_{2,xxx} + \alpha u_2 u_{1,xxx} + u_{1,xxxxx} + 2\gamma u_1 u_2 u_{2,x} + \gamma u_2^2 u_{1,x} + \beta u_{1,x} u_{2,xx} + \beta u_{2,x} u_{1,xx} + u_{1,t} = 0, \quad (12)$$

ϕ^0 : u_2 needs to satisfy the original equation, i.e.,

$$u_{2,xxxxx} + \beta u_{2,x} u_{2,xx} + \alpha u_2 u_{2,xxx} + u_{2,t} + \gamma u_2^2 u_{2,x} = 0 \quad (13)$$

The set of (5) and (6 - 13) constitutes an *auto-Bäcklund transformation*, if the set is solvable with respect to $\phi(x, t)$, $u_0(x, t)$, $u_1(x, t)$ and $u_2(x, t)$ [15, 16]. Equation (6) brings out three possibilities for $u_0(x, t)$:

$$u_0^I = 0, \quad u_0^{II} = 3 \frac{(-\beta - 2\alpha + \mathcal{H}) \phi_x^2}{\gamma}, \quad u_0^{III} = 3 \frac{(-\beta - 2\alpha - \mathcal{H}) \phi_x^2}{\gamma}, \quad (14)$$

where $\mathcal{H} \equiv \sqrt{\beta^2 + 4\beta\alpha + 4\alpha^2 - 40\gamma}$. Some complicated symbolic manipulations are required to find general solutions for the remaining $u_1(x, t)$ and $u_2(x, t)$. However, due to increasing complexities in the symbolic computations, a long CPU time is required. Thus, in the following, in order to shorten the computation time, we constrain the model parameters by requiring

$$\mathcal{H} = 0 \implies \alpha = -\beta/2 + \sqrt{10\gamma} \quad \text{or} \quad \alpha = -\beta/2 - \sqrt{10\gamma}. \quad (15)$$

In the rest of the paper, we consider a solution family for the case of the non-trivial solution u_0^{III} with the first constraint $\alpha = -\beta/2 + \sqrt{10\gamma}$. Of course, other classes of solitary-wave solutions can be found from u^{III} and the second constraint on α, β, γ .

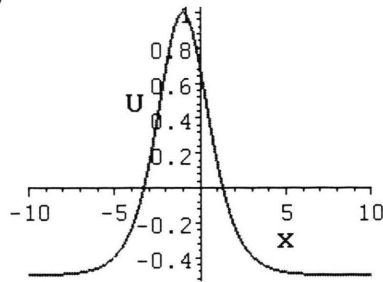
After substituting u_0^{III} and α into (7), we obtain

$$u_1(x, t) = \frac{12\phi_{xx}(5\beta + 2\sqrt{10\gamma})}{\beta\sqrt{10\gamma} + 4\gamma}. \quad (16)$$

Subsequently, we get the following solution for $u_2(x, t)$ from (9):

$$u_2(x, t) = \frac{-32\gamma\phi_x\phi_{xxx} + 15\beta^2\phi_{xx}^2 - 20\beta^2\phi_x\phi_{xxx} - 4\phi_x\phi_{xxx}\sqrt{10\gamma}\beta + 24\phi_{xx}^2\gamma}{\phi_x^2(\sqrt{10\gamma}\beta^2 + 8\gamma\beta + 16\gamma^{3/2})}. \quad (17)$$

(a)



(b)

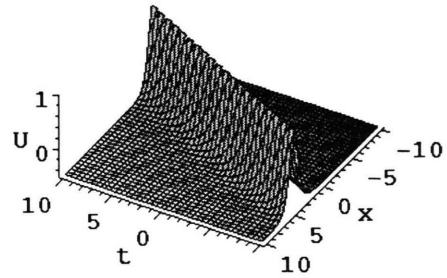


Fig. 1. (a) Typical sech²-shaped solitary-wave solution $u(x, 0)$ in (21) with parameters $A = 1, B = 1, C_1 = 1, \beta = -2, \gamma = 10$, and $\alpha = -\beta/2 + \sqrt{10\gamma} = 11$. (b) $u(x, t)$ shows the solitary wave property that the amplitude becomes finite as $|x|$ approaches infinity.

Thus, we are able to find a class of analytical solutions $u(x, t)$ in terms of $u_0^{\text{II}}, u_1, u_2$ in (14 - 17) with an arbitrary function $\phi(x, t)$ constrained by the remaining equations (10 - 13). We note that, once a Bäcklund transformation is discovered and a set of “seed” solutions is given, one will be able to find an infinite number of solutions by repeated applications of the transformation, i. e., to generate a hierarchy of solutions with increasing complexity. In the rest of the paper we will find a family of some exact analytical solutions of (1).

Sample Solution

A trial solution

$$\phi(x, t) = 1 + e^{\mathcal{Q}(x, t)}, \text{ where } \mathcal{Q}(x, t) = A(t)x + B(t) \quad (18)$$

is substituted into the remaining constraint equations (10 - 13). From (10), we find

$$\phi^{-3} \& x : A(t)' = 0 \Rightarrow A(t) = A = \text{constant}. \quad (19)$$

Then, we obtain $B(t)'$ from the terms of ϕ^{-3} and integrate it over t to get

$$B(t) = \frac{-\mathcal{R} \cdot t - \mathcal{S}}{4\mathcal{T}},$$

$$\begin{aligned} \mathcal{R} \equiv & 10500 A^5 \sqrt{10} \beta^6 \gamma - 62720 A^5 \sqrt{10} \gamma^3 \beta^2 \\ & + 13750 A^5 \beta^7 \sqrt{\gamma} - 97280 A^5 \gamma^{7/2} \beta \\ & - 201600 A^5 \beta^3 \gamma^{5/2} + 14000 A^5 \beta^5 \gamma^{3/2} \\ & - 28000 A^5 \sqrt{10} \beta^4 \gamma^2 - 6144 A^5 \sqrt{10} \gamma^4 \\ & + 625 A^5 \beta^8 \sqrt{10}, \end{aligned}$$

$$\begin{aligned} \mathcal{S} \equiv & -224000 C_1 \beta^3 \gamma^{5/2} - 2500 C_1 \beta^7 \sqrt{\gamma} \\ & - 84000 C_1 \beta^5 \gamma^{3/2} - 71680 C_1 \gamma^{7/2} \beta \\ & - 53760 C_1 \sqrt{10} \gamma^3 \beta^2 - 4096 C_1 \sqrt{10} \gamma^4 \\ & - 7000 C_1 \sqrt{10} \beta^6 \gamma - 56000 C_1 \sqrt{10} \beta^4 \gamma^2, \end{aligned}$$

$$\begin{aligned} \mathcal{T} \equiv & 1024 \gamma^4 \sqrt{10} + 1750 \sqrt{10} \beta^6 \gamma + 14000 \sqrt{10} \beta^4 \gamma^2 \\ & + 13440 \sqrt{10} \gamma^3 \beta^2 + 56000 \beta^3 \gamma^{5/2} + 625 \beta^7 \sqrt{\gamma} \\ & + 21000 \beta^5 \gamma^{3/2} + 17920 \gamma^{7/2} \beta, \end{aligned} \quad (20)$$

where C_1 is the constant of integration. Combining all terms, we find a family of analytical solutions of (1) as

$$\begin{aligned} u(x, t) = & 3 \frac{(-\beta - 2\alpha + \mathcal{H}) \phi_x^2}{\gamma} \cdot \frac{1}{(1 + e^{\mathcal{Q}(x, t)})^2} \\ & + \frac{12 \phi_{xx} (5\beta + 2\sqrt{10}\gamma)}{\beta \sqrt{10}\gamma + 4\gamma} \cdot \frac{1}{1 + e^{\mathcal{Q}(x, t)}} \\ & - \left[32 \gamma \phi_x \phi_{xxx} - 15 \beta^2 \phi_{xx}^2 + 20 \beta^2 \phi_x \phi_{xxx} \right. \\ & \quad \left. + 4 \phi_x \phi_{xxx} \sqrt{10}\gamma \beta - 24 \phi_{xx}^2 \gamma \right] \\ & \quad \left/ \left[\phi_x^2 (\sqrt{10}\gamma \beta^2 + 8\gamma \beta + 16\gamma^{3/2}) \right] \right. \end{aligned} \quad (21)$$

with the trial function $\phi(x, t)$ in (18) with $\mathcal{Q}(x, t)$ in (19, 20).

In the following we show that our solutions indeed have solitary-wave properties by presenting some figures from the family: We choose, as an example, a set of arbitrary constants; $A = 1, B = 1$, and $C_1 = 1$

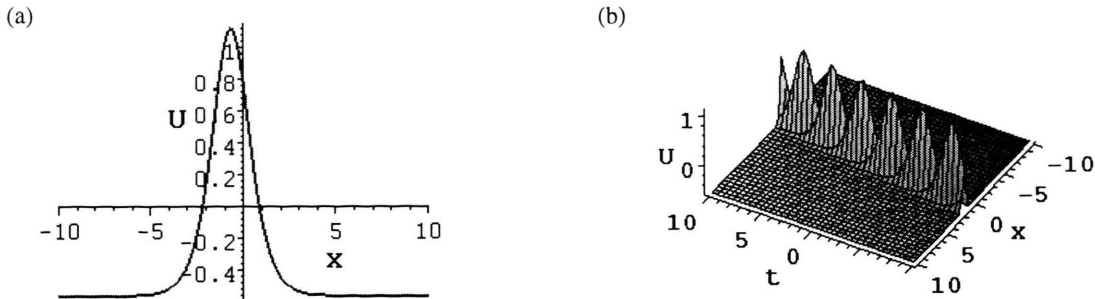


Fig. 2. (a) Sech^2 -shaped solitary-wave solution $u(x, 0)$ with parameters $A = 1$, $B = 1$, $C_1 = 1$, $\beta = 2$, $\gamma = 40$, and $\alpha = 19$. (b) $u(x, t)$ shows the breather solitary-wave solution behavior.

with the constraint $\alpha = -\beta/2 + \sqrt{10\gamma}$. Figures 1 and 2 correspond to $(\beta = -2, \gamma = 10, \alpha = 11)$ and $(\beta = 2, \gamma = 40, \alpha = 19)$, respectively. Firstly, we note that $u(x, 0)$ in Figs. 1(a) and 2(a) are both $\text{sech}(x)^2$ -shaped solutions. From Figs. 1(b) and 2(b) we understand that in both cases the solutions have solitary wave property, i. e., $u(x, t)$ tends to a finite value as $|x|$ approaches infinity. Interestingly, Fig. 2(b) shows a breather solitary-wave solution.

To sum up, with symbolic computations and the truncated Painlevé expansion analysis, we showed that Bäcklund transformations exist for the generalized fifth-order non-integrable nonlinear evolution

equation. We found a class of analytical solutions $u(x, t)$ to (1) in terms of $u_0^{\text{II}}, u_1, u_2$ in (14 - 17) with an arbitrary function $\phi(x, t)$ constrained by (10 - 13). A sample solution family for $u(x, t)$ was found in Eq.(21) with the constraint $\alpha = -\beta/2 + \sqrt{10\gamma}$. More solitary-wave families can be found from $u^{\text{III}}(x, t)$ and other constraints on α, β, γ in (15).

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